

SOME q -ANALOGS OF CONGRUENCES FOR CENTRAL BINOMIAL SUMS

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ABSTRACT. We establish q -analogs for four congruences involving central binomial coefficients. The q -identities necessary for this purpose are shown via the q -WZ method.

1. INTRODUCTION

Recently, a number of papers have appeared concerning congruences for central binomial sums (see through the references). Here we would like to draw the attention to one aspect of the matter which has been partly neglected so far: q -analogs. In [5, 11], the authors identified a first group of such congruences which have a q -counterpart. Among them we mention: for any prime $p > 2$

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{2k}{k} &\equiv \left(\frac{p}{3}\right) \pmod{p^2}, & \sum_{k=0}^{p-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q &\equiv \left(\frac{p}{3}\right) q^{\lfloor \frac{p^2}{3} \rfloor} \pmod{[p]_q}, \\ \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} &\equiv \left(\frac{p}{5}\right) \pmod{p}, & \sum_{k=0}^{p-1} (-1)^k q^{-\binom{k+1}{2}} \begin{bmatrix} 2k \\ k \end{bmatrix}_q &\equiv \left(\frac{p}{5}\right) q^{-\lfloor \frac{p^4}{5} \rfloor} \pmod{[p]_q}, \\ \sum_{k=0}^{p-1} \frac{1}{2^k} \binom{2k}{k} &\equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}, & \sum_{k=1}^{p-1} \frac{q^k}{(-q; q)_k} \begin{bmatrix} 2k \\ k \end{bmatrix}_q &\equiv (-1)^{\frac{p-1}{2}} q^{\lfloor \frac{p^2}{4} \rfloor} \pmod{[p]_q}, \end{aligned}$$

where (\cdot) denotes the Legendre symbol. It has been conjectured in [5] that the first q -congruence holds modulo $[p]_q^2$, and we claim that the same can be said for the third one. However, in this short note, we are not going to refine these q -congruences. Instead, we will present a few more examples of this phenomena. More precisely we show that the congruences

$$\begin{aligned} (1) \quad & \sum_{k=1}^{p-1} \frac{(1/2)^k}{k} \binom{2k}{k} \equiv -\frac{3}{2} \sum_{k=1}^{p-1} \frac{(-1/2)^k}{k} \binom{2k}{k}^{-1} \equiv Q_p(2) \pmod{p}, \\ (2) \quad & \sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} \equiv 0 \pmod{p^2}, \\ (3) \quad & \frac{5}{2} \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv -\sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^3}, \end{aligned}$$

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where $p > 5$ is a prime and $Q_p(2) = (2^{p-1} - 1)/p$ is the usual Fermat's quotient, have as q -analogs respectively

$$(4) \quad \sum_{k=1}^{p-1} \frac{q^k}{(-q; q)_k [k]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv -\frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^k (1 + q^k + q^{2k})}{(-q; q)_k [k]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} \equiv Q_p(2; q) \pmod{[p]_q},$$

$$(5) \quad \sum_{k=1}^{p-1} \frac{(1 + q^k + q^{2k}) q^{-\binom{k}{2}}}{(1 + q^k)^2 [k]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv \frac{[p]_q (p^2 - 1)(1 - q)^2}{24} \pmod{[p]_q^2},$$

$$(6) \quad \sum_{k=1}^{p-1} \frac{(-1)^k (1 + 3q^k + q^{2k}) q^{-\binom{k}{2}}}{(1 + q^k) [k]_q^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv -\sum_{k=1}^{p-1} \frac{q^k}{[k]_q^2} - \frac{[p]_q^2 (p^4 - 1)(1 - q)^4}{240} \pmod{[p]_q^3}.$$

where $Q_p(2; q) = ((-q; q)_{p-1} - 1)/[p]_q$. Proofs of (1), (2), (3) can be found in [12, Theorem 3.1] ((2) appeared first in [10]).

We are optimistically hopeful that there are plenty of interesting q -analogs to discover. For example, recently in [8], the authors proved that for $0 < q < 1$

$$\sum_{k=1}^{\infty} \frac{(1 + 2q^k) q^{k^2}}{[k]_q^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} = \sum_{k=1}^{\infty} \frac{q^k}{[k]_q^2}.$$

By letting $q \rightarrow 1$, it gives a well known series identity which then happened to have a congruence version: in [12] we showed that for any prime $p > 3$

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \binom{2k}{k}^{-1} \equiv -\frac{1}{6} \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^3}.$$

Is there a q -analog for the above congruence?

2. NOTATIONS A PRELIMINARY RESULTS

The first two results of this section yield a family of q -analogs of the classical congruence for the harmonic sums: for any prime $p > d + 2$ where d is a positive integer,

$$H_{p-1}(d) := \sum_{k=1}^{p-1} \frac{1}{k^d} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } d \text{ is odd,} \\ 0 \pmod{p} & \text{if } d \text{ is even.} \end{cases}$$

This family depends on two integer parameters a, b and it concerns the sum

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^d}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}.$$

Several special cases have already been discussed by numerous authors (see [3, 4, 7, 9]). In particular, K. Dilcher found in [4] a determinant expression in the case when $a = 1$ and $b \in \{0, 1\}$. We point out that, in this paper, two rational functions in q are congruent modulo $[p]_q^r$ for $r \geq 1$ if the numerator of their difference is congruent to 0 modulo $[p]_q^r$ in the polynomial ring $\mathbb{Z}[q]$ and the denominator is relatively prime to $[p]_q$.

Theorem 2.1. *For any prime $p > 2$, if a, b, d are integers such that $a, d > 0$, $b \geq 0$ and $\gcd(a, p) = 1$ then*

$$(7) \quad \sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^d} \equiv \frac{(1-q)^d}{p^d} \left((-1)^d p \sum_{s=0}^{d-1} c_s \binom{r_0 + sp}{d} - \sum_{s=0}^d (-1)^s \binom{d}{s} \binom{sp}{2d} \right) \pmod{[p]_q}$$

where $r_0 \equiv -b/a \pmod{p}$ such that $r_0 \in \{0, 1, \dots, p-1\}$ and

$$c_s = \sum_{k=0}^s (-1)^{s-k} \binom{r_0 + kp + d - 1}{d-1} \binom{d}{s-k}.$$

Proof. Let q be a p -root of unity such that $q \neq 1$. Since

$$\sum_{k=1}^{p-1} q^{k(aj+b)} = -1 + \begin{cases} p & \text{if } p \mid (aj+b), \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{q^{bk}}{(1 - q^{ak}z)^d} &= \sum_{k=1}^{p-1} q^{bk} \sum_{j \geq 0} \binom{j+d-1}{d-1} q^{akj} z^j \\ &= \sum_{j \geq 0} \binom{j+d-1}{d-1} z^j \sum_{k=1}^{p-1} q^{k(aj+b)} \\ &= p \sum_{s \geq 0} \binom{r_0 + sp + d - 1}{d-1} z^{r_0+sp} - \frac{1}{(1-z)^d} \\ &= \frac{p \sum_{s=0}^{d-1} c_s z^{r_0+sp}}{(1-z^p)^d} - \frac{1}{(1-z)^d}. \end{aligned}$$

Let $z = 1 + w$, then

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{q^{bk}}{(1 - q^{ak})^d} &= \lim_{w \rightarrow 0} \frac{p \sum_{s=0}^{d-1} c_s (1+w)^{r_0+sp} - \left(\frac{1-(1+w)^p}{-w} \right)^d}{(1 - (1+w)^p)^d} \\ &= \lim_{w \rightarrow 0} \frac{p \sum_{s=0}^{d-1} c_s \binom{r_0+sp}{d} w^d + o(w^d) - (-1)^d \sum_{s=0}^d (-1)^s \binom{d}{s} \binom{sp}{2d} w^d + o(w^d)}{(-pw + o(w))^d} \\ &= \frac{1}{p^d} \left((-1)^d p \sum_{s=0}^{d-1} c_s \binom{r_0 + sp}{d} - \sum_{s=0}^d (-1)^s \binom{d}{s} \binom{sp}{2d} \right). \end{aligned}$$

□

The following special cases are worth mentioning. By letting $d = 1, 2, 3$ in (7), we obtain these q -congruences which hold modulo $[p]_q$:

$$(8) \quad \sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q} \equiv (1-q) \left(\frac{p-1}{2} - r_0 \right),$$

$$(9) \quad \sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^2} \equiv (1-q)^2 \left(-\frac{(p-1)(p-5)}{12} + \frac{r_0(p-2-r_0)}{2} \right),$$

$$(10) \quad \sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^3} \equiv (1-q)^3 \left(-\frac{(p-1)(p-3)}{8} - \frac{r_0(p^2-9p+12-3r_0(p-3)+2r_0^2)}{12} \right).$$

Theorem 2.2. *Let b, \bar{b}, a, d be non-negative integers such that $ad = b + \bar{b} > 0$. Then for any prime $p > 2$ such that $\gcd(a, p) = 1$,*

$$(11) \quad \sum_{k=1}^{p-1} \frac{(-1)^{d-1} q^{bk} + q^{\bar{b}k}}{[ak]_q^d} \equiv b(1-q)[p]_q \sum_{k=1}^{p-1} \frac{q^{\bar{b}k}}{[ak]_q^d} - ad[p]_q \sum_{k=1}^{p-1} \frac{q^{\bar{b}k}}{[ak]_q^{d+1}} \pmod{[p]_q^2}$$

and

$$(12) \quad \sum_{k=1}^{p-1} \frac{(-1)^k ((-1)^d q^{bk} + q^{\bar{b}k})}{[ak]_q^d} \equiv b(1-q)[p]_q \sum_{k=1}^{p-1} \frac{(-1)^k q^{\bar{b}k}}{[ak]_q^d} - ad[p]_q \sum_{k=1}^{p-1} \frac{(-1)^k q^{\bar{b}k}}{[ak]_q^{d+1}} \pmod{[p]_q^2}.$$

Moreover, let $b_1, \bar{b}_1, b_2, \bar{b}_2$ be non-negative integers.

If $d_1 = b_1 + \bar{b}_1 > 0$ and $d_2 = b_2 + \bar{b}_2 > 0$ then

$$(13) \quad \sum_{1 \leq j < k \leq p-1} \frac{q^{b_1 j + b_2 k} + (-1)^{d_1 + d_2} q^{\bar{b}_1 j + \bar{b}_2 k}}{[j]_q^{d_1} [k]_q^{d_2}} \equiv \sum_{k=1}^{p-1} \frac{q^{b_1 j}}{[j]_q^{d_1}} \cdot \sum_{k=1}^{p-1} \frac{q^{b_2 k}}{[k]_q^{d_2}} - \sum_{k=1}^{p-1} \frac{q^{(b_1 + b_2)k}}{[k]_q^{d_1 + d_2}} \pmod{[p]_q}.$$

Proof. As regards (11) and (12), it suffices to note that

$$(14) \quad \begin{aligned} \frac{(-1)^d q^{b(p-k)}}{[ap-k]_q^d} &= \frac{(-1)^d q^{b(p-k) + adk}}{([ap]_q - [ak]_q)^d} = \frac{q^{bp + \bar{b}k}}{[ak]_q^d (1 - [ap]_q/[ak]_q)^d} \\ &\equiv \frac{q^{\bar{b}k} (1 - b(1-q)[p]_q)}{[ak]_q^d} \left(1 + d \frac{[p]_q}{[ak]_q} \right) \\ &\equiv \frac{q^{\bar{b}k}}{[ak]_q^d} - \frac{b(1-q)[p]_q q^{\bar{b}k}}{[ak]_q^d} + \frac{ad[p]_q q^{\bar{b}k}}{[ak]_q^{d+1}} \pmod{[p]_q^2}. \end{aligned}$$

Moreover

$$\sum_{k=1}^{p-1} \frac{q^{b_1 j}}{[j]_q^{d_1}} \cdot \sum_{k=1}^{p-1} \frac{q^{b_2 k}}{[k]_q^{d_2}} - \sum_{k=1}^{p-1} \frac{q^{(b_1 + b_2)k}}{[k]_q^{d_1 + d_2}} = \sum_{1 \leq j < k \leq p-1} \frac{q^{b_1 j + b_2 k}}{[j]_q^{d_1} [k]_q^{d_2}} + \sum_{1 \leq k < j \leq p-1} \frac{q^{b_1 j + b_2 k}}{[j]_q^{d_1} [k]_q^{d_2}}$$

and by (14) we get

$$\sum_{1 \leq k < j \leq p-1} \frac{q^{b_1 j + b_2 k}}{[j]_q^{d_1} [k]_q^{d_2}} = \sum_{1 \leq j < k \leq p-1} \frac{q^{b_1(p-j) + b_2(p-k)}}{[p-j]_q^{d_1} [p-k]_q^{d_2}} \equiv \sum_{1 \leq j < k \leq p-1} \frac{(-1)^{d_1 + d_2} q^{\bar{b}_1 j + \bar{b}_2 k}}{[j]_q^{d_1} [k]_q^{d_2}} \pmod{[p]_q}.$$

Hence the proof of (13) is complete. \square

By letting $a = 1$, $d = 1$ and $b = 0$ in (11), and by using (7), we easily find [9, Theorem 1]: for any prime $p > 3$:

$$\sum_{k=1}^{p-1} \frac{1}{[k]_q} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{1-q^k}{[k]_q} \frac{1}{2} \sum_{k=1}^{p-1} \frac{1+q^k}{[k]_q} \equiv \frac{(1-q)(p-1)}{2} + \frac{(p^2-1)(1-q)^2[p]_q}{24} \pmod{[p]_q^2}.$$

In a similar way, for $a = 1$, $d = 3$ and $b = 1$, (11) yields

$$(15) \quad \sum_{k=1}^{p-1} \frac{q^k + q^{2k}}{[k]_q^3} \equiv (1-q)[p]_q \sum_{k=1}^{p-1} \frac{q^{2k}}{[k]_q^3} - 3[p]_q \sum_{k=1}^{p-1} \frac{q^{2k}}{[k]_q^4} \equiv -\frac{[p]_q(1-q)^4(p^4-1)}{240} \pmod{[p]_q^2}.$$

In order to show the q -congruences stated in the introduction, we need *suitable* q -identities. Such identities are not easy to find, but ones they are guessed correctly hopefully they can be proved via the q -WZ method (see for example [6, 13]).

For $n \geq k \geq 0$, a pair $(F(n, k), G(n, k))$ is called q -WZ pair if

$$F(n+1, k)/F(n, k), \quad F(n, k+1)/F(n, k), \quad G(n+1, k)/G(n, k), \quad G(n, k+1)/G(n, k)$$

are all rational functions of q^n and q^k , and

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k).$$

Let

$$S(n) = \sum_{k=0}^{N-1} F(n, k)$$

then

$$\begin{aligned} S(n+1) - S(n) &= F(n+1, n) + \sum_{k=0}^{n-1} (F(n+1, k) - F(n, k)) \\ &= F(n+1, n) + \sum_{k=0}^{n-1} (G(n, k+1) - G(n, k)) \\ &= F(n+1, n) + G(n, n) - G(n, 0) \end{aligned}$$

and, by summing over n from 0 to $N-1$, we get the identity

$$(16) \quad \sum_{k=0}^{N-1} F(N, k) = \sum_{n=0}^{N-1} (F(n+1, n) + G(n, n)) - \sum_{n=0}^{N-1} G(n, 0)$$

which can be considered as the finite form of [6, Theorem 7.3].

The q -identities we are interested in involves *Gaussian q -binomial coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} (q; q)_n (q; q)_k^{-1} (q; q)_{n-k}^{-1} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

where $(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$ (note that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q). The next lemma allows us to reduce a special class of q -binomial coefficients modulo a power of $[p]_q$.

Lemma 2.3. *Let p be a prime and let a be a positive integer. For $k = 1, \dots, p-1$ we have*

$$(17) \quad \left[\begin{matrix} ap-1 \\ k \end{matrix} \right]_q \equiv (-1)^k q^{-\binom{k+1}{2}} \left(1 - a[p]_q \sum_{j=1}^k \frac{1}{[j]_q} \right) \pmod{[p]_q^2},$$

$$(18) \quad \left[\begin{matrix} p-1+k \\ k \end{matrix} \right]_q \equiv \frac{[p]_q}{[k]_q} \left(1 + [p]_q \sum_{j=1}^{k-1} \frac{q^j}{[j]_q} \right) \pmod{[p]_q^3},$$

$$(19) \quad \left[\begin{matrix} p-1+k \\ k \end{matrix} \right]_q \left[\begin{matrix} p-1 \\ k \end{matrix} \right]_q^{-1} \equiv \frac{(-1)^k q^{\binom{k+1}{2}} [p]_q}{[k]_q} \left(1 + \frac{[p]_q}{[k]_q} + [p]_q \sum_{j=1}^{k-1} \frac{1+q^j}{[j]_q} \right) \pmod{[p]_q^3}.$$

Proof. Since $[ap]_q \equiv a[p]_q \pmod{[p]_q^2}$, it follows that

$$\left[\begin{matrix} ap-1 \\ k \end{matrix} \right]_q = (-1)^k q^{-\binom{k+1}{2}} \prod_{j=1}^k \left(1 - \frac{[ap]_q}{[j]_q} \right) \equiv (-1)^k q^{-\binom{k+1}{2}} \left(1 - a[p]_q \sum_{j=1}^k \frac{1}{[j]_q} \right) \pmod{[p]_q^2}.$$

Moreover

$$\left[\begin{matrix} p-1+k \\ k \end{matrix} \right]_q = \frac{[p]_q}{[k]_q} \prod_{j=1}^{k-1} \left(1 + \frac{q^j [p]_q}{[j]_q} \right) \equiv \frac{[p]_q}{[k]_q} \left(1 + [p]_q \sum_{j=1}^{k-1} \frac{q^j}{[j]_q} \right) \pmod{[p]_q^3}.$$

Congruences (17) and (18) easily yield (19). \square

It should be noted that when p is an odd prime, by using (18) for $k = p-1$, we recover the q -congruence [3, (3.2)]:

$$(20) \quad \left[\begin{matrix} ap-1 \\ p-1 \end{matrix} \right]_q \equiv q^{-\binom{p}{2}} \left(1 - a[p]_q \sum_{j=1}^{p-1} \frac{1}{[j]_q} \right) \equiv q^{-\binom{p}{2}} \left(1 - \frac{a[p]_q(p-1)(1-q)}{2} \right) \equiv q^{(a-1)\binom{p}{2}} \pmod{[p]_q^2}.$$

3. PROOF OF (4)

By [2, (5.17)] (see [5, (4.1)] for a generalization), if n is odd then by

$$\sum_{k=0}^n \frac{(-1)^{n-k} q^{\binom{n-k}{2}}}{(-q; q)_k} \left[\begin{matrix} n \\ k \end{matrix} \right]_q \left[\begin{matrix} 2k \\ k \end{matrix} \right]_q = 0.$$

Hence for $n = p$ we have that

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1} q^{\binom{p-k}{2} - \binom{p}{2}}}{(-q; q)_k [k]_q} \left[\begin{matrix} p-1 \\ k-1 \end{matrix} \right]_q \left[\begin{matrix} 2k \\ k \end{matrix} \right]_q = \frac{1}{(-q; q)_{p-1} [p]_q} \left((-q; q)_{p-1} - q^{-\binom{p}{2}} \left[\begin{matrix} 2p-1 \\ p-1 \end{matrix} \right]_q \right).$$

By (17) and (20) we get

$$\sum_{k=1}^{p-1} \frac{q^k}{(-q; q)_k [k]_q} \left[\begin{matrix} 2k \\ k \end{matrix} \right]_q \equiv \sum_{k=1}^{p-1} \frac{q^{-pk+k}}{(-q; q)_k [k]_q} \left[\begin{matrix} 2k \\ k \end{matrix} \right]_q \equiv \frac{(-q; q)_{p-1} - 1}{(-q; q)_{p-1} [p]_q} \equiv Q_p(2; q) \pmod{[p]_q},$$

and the first congruence is proved.

As regards the second one, we take

$$F(n, k) = \frac{(-1)^k}{(-q; q)_n [k+1]_q} \left[\begin{matrix} n+k+1 \\ k+1 \end{matrix} \right]_q^{-1} \quad \text{and} \quad G(n, k) = \frac{q^{n+1} F(n, k)}{1 + q^{n+1}}.$$

This pair can be found in [1][Subsection 2.1] in connection with the irrationality proof of the q -series

$$\text{Ln}_q(2) := \sum_{k=1}^{\infty} \frac{(-1)^k}{q^n - 1}$$

for $|q| \notin \{0, 1\}$. Hence by (16) we obtain the identity

$$(21) \quad \sum_{k=1}^n \frac{(-1)^k (1 + q^k + q^{2k})}{(-q; q)_k [k]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} = \frac{1}{(-q; q)_n} \sum_{k=1}^n \frac{(-1)^k}{[k]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q^{-1} - \sum_{k=1}^n \frac{q^k}{(-q; q)_k [k]_q}$$

Let $n = p - 1$. Now by (17) and [7, (1.5)]

$$\begin{aligned} \frac{1}{(-q; q)_{p-1}} \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q^{-1} &\equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{[p]_q} \left(1 - [p]_q \sum_{j=1}^{k-1} \frac{q^j}{[j]_q} \right) \\ &\equiv - \sum_{k=1}^{p-1} (-1)^k \sum_{j=1}^{k-1} \frac{q^j}{[j]_q} \equiv - \sum_{j=1}^{p-1} \frac{q^{2j-1}}{[2j-1]_q} \\ &\equiv \frac{(p-1)(1-q)}{2} - \sum_{k=1}^{p-1} \frac{1}{[j]_q} + \sum_{k=1}^{(p-1)/2} \frac{1}{[2j]_q} \equiv -Q_p(2; q) \pmod{[p]_q}. \end{aligned}$$

By the q -binomial theorem,

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1} (x - q^j) = x^n$$

and for $n = p$, $x = -1$ together with (17), we get

$$\sum_{k=1}^{p-1} \frac{q^{-\binom{k}{2}} (-q; q)_{k-1}}{[k]_q} \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1} (-q; q)_{k-1}}{[k]_q} \begin{bmatrix} p-1 \\ k-1 \end{bmatrix}_q = -Q_p(2; q).$$

Hence (see the *dual* congruence [7, (5.4)])

$$\sum_{k=1}^{p-1} \frac{q^k}{(-q; q)_k [k]_q} \equiv \sum_{k=1}^{p-1} \frac{q^{p-k}}{(-q; q)_{p-k} [p-k]_q} \equiv \sum_{k=1}^{p-1} \frac{q^{-\binom{k}{2}} (-q; q)_{k-1}}{[k]_q} \equiv -Q_p(2; q) \pmod{[p]_q}$$

where we used

$$[p-k]_q = -q^{-k} [k]_q \quad \text{and} \quad (-q; q)_{p-k}^{-1} \equiv q^{-\binom{k}{2}} (-q; q)_{k-1} \pmod{[p]_q}.$$

Therefore, by identity (21),

$$\sum_{k=1}^{p-1} \frac{(-1)^k (1 + q^k + q^{2k})}{(-q; q)_k [k]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} \equiv -2Q_p(2; q) \pmod{[p]_q}$$

and we are done.

4. PROOF OF (5)

Let

$$F(n, k) = \frac{q^{-(\binom{k+1}{2})}}{[k+1]_q} \begin{bmatrix} n+k+2 \\ n+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q^{-1}$$

and

$$G(n, k) = -\frac{q^{n+2}(1-q^{k+1})(1-q^{n+1-k})F(n, k)}{(1+q^{n+2})(1-q^{n+2})^2}$$

then (16) gives the identity

$$(22) \quad \sum_{k=1}^n \frac{(1+q^k+q^{2k})q^{-(\binom{k}{2})}}{(1+q^k)^2[k]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q = \sum_{k=1}^n \frac{q^{-(\binom{k}{2})}}{[k]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q^{-1} - \sum_{k=1}^n \frac{q^k}{[2k]_q}.$$

Let $n = p-1$ then by (19) and (12), we obtain

$$\sum_{k=1}^{p-1} \frac{q^{-(\binom{k}{2})}}{[k]_q} \begin{bmatrix} p-1+k \\ k \end{bmatrix}_q \begin{bmatrix} p-1 \\ k \end{bmatrix}_q^{-1} \equiv [p]_q \sum_{k=1}^{p-1} \frac{(-1)^k q^k}{[k]_q^2} \equiv 0 \pmod{[p]_q^2}.$$

Moreover, (11) and (9) implies that

$$\sum_{k=1}^{p-1} \frac{q^k}{[2k]_q} \equiv -[p]_q \sum_{k=1}^{p-1} \frac{q^k}{[2k]_q^2} \equiv -\frac{[p]_q(p^2-1)(1-q)^2}{24} \pmod{[p]_q^2},$$

and (5) follows easily from (22).

Note that by letting $q \rightarrow 1$ in (22) we obtain the identity

$$\frac{3}{4} \sum_{k=1}^n \frac{1}{k} \binom{2k}{k} = \sum_{k=1}^n \frac{1}{k} \binom{n+k}{k} \binom{n}{k}^{-1} - \frac{1}{2} \sum_{k=1}^n \frac{1}{k}.$$

which can be exploited to prove an improvement of [12, Theorem 4.2]:

$$\sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} \equiv -\frac{8}{3} H_{p-1}(1) + 2p^4 B_{p-5} \pmod{p^5}$$

for any prime $p > 3$.

5. PROOF OF (6)

By taking

$$F(n, k) = \frac{(-1)^k q^{-(\binom{k+1}{2})} (1+q^{k+1})}{[k+1]_q^2} \begin{bmatrix} n+k+2 \\ n+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q^{-1}$$

and

$$G(n, k) = \frac{q^{n+2}(1-q^{k+1})^2(1-q^{n+1-k})F(n, k)}{(1+q^{k+1})(1-q^{n+2})^3},$$

(16) yields the identity

$$(23) \quad \sum_{k=1}^n \frac{(-1)^k (1+3q^k+q^{2k})q^{-(\binom{k}{2})}}{(1+q^k)[k]_q^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_q = \sum_{k=1}^n \frac{(-1)^k (1+q^k)q^{-(\binom{k}{2})}}{[k]_q^2} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q^{-1} - \sum_{k=1}^n \frac{q^k}{[k]_q^2}.$$

Let $n = p - 1$. Now by (19)

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(-1)^k (1+q^k) q^{-\binom{k}{2}}}{[k]_q^2} \begin{bmatrix} p-1+k \\ k \end{bmatrix}_q \begin{bmatrix} p-1 \\ k \end{bmatrix}_q^{-1} &\equiv [p]_q \sum_{k=1}^{p-1} \frac{q^k + q^{2k}}{[k]_q^3} + [p]_q^2 \sum_{k=1}^{p-1} \frac{q^k + q^{2k}}{[k]_q^4} \\ &\quad + [p]_q^2 \sum_{1 \leq j < k \leq p-1} \frac{(1+q^j)(q^k + q^{2k})}{[j]_q [k]_q^3} \pmod{[p]_q^3}. \end{aligned}$$

Then the q -congruence (6) follows from (23) by using (15) and

$$\sum_{k=1}^{p-1} \frac{q^k + q^{2k}}{[k]_q^4} \equiv - \sum_{1 \leq j < k \leq p-1} \frac{(1+q^j)(q^k + q^{2k})}{[j]_q [k]_q^3} \equiv \frac{(1-q)^4 (p^2-1)(p^2-4)}{360} \pmod{[p]_q},$$

which is a straightforward application of (7) and (13).

REFERENCES

- [1] T. Amdeberhan, D. Zeilberger, q -Apéry irrationality proofs by q -WZ pairs, Adv. in Appl. Math. **20** (1998), 275–283.
- [2] G. E. Andrews, *Applications of basic hypergeometric functions*, SIAM Rev. **16** (1974), 441–484.
- [3] G. E. Andrews, q -Analogues of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher, Discrete Math. **204** (1999), 15–25.
- [4] K. Dilcher, *Determinant expressions for q -harmonic congruences and degenerate Bernoulli numbers*, Electron. J. Combin. **15** (2008), 1–18.
- [5] V. J. W. Guo, J. Zeng, *Some congruences involving central q -binomial coefficients*, Adv. in Appl. Math. **45** (2010), 303–316.
- [6] M. Mohammed, *The q -Markov-WZ method*, Ann. Comb. **9** (2005), 205–221.
- [7] H. Pan, *A q -analogue of Lehmer’s congruence*, Acta Arith. **128** (2007), 303–318.
- [8] Kh. Hessami Pilehrood, T. Hessami Pilehrood, *A q -analogue of the Bailey-Borwein-Bradley identity*, J. Symbolic Comput. **46** (2011), 699–711.
- [9] L. L. Shi, H. Pan, *A q -analogue of Wolstenholme’s harmonic series congruence*, Amer. Math. Monthly **114** (2007), 529–531.
- [10] Z. W. Sun, R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. **45** (2010), 125–148.
- [11] R. Tauraso, *q -Analogues of some congruences involving Catalan numbers*, to appear in Adv. in Appl. Math.
- [12] R. Tauraso, *More congruences for central binomial coefficients*, J. Number Theory **130** (2010), 2639–2649.
- [13] D. Zeilberger, *Closed form (pun intended!)*, Contemp. Math. **143** (1993), 579–607.

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